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Discrete Mathematics 85 (1990) 215–219
North-Holland 

NOTE

# REGULAR FACTORS OF LINE GRAPHS

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Received 25 January 1988
Revised 13 October 1988

We show that if $m \geq 2$ is an even integer and $G$ is a graph such that $d_G(v) \geq m+1$ for all vertices $v$ in $G$, then the line graph $L(G)$ of $G$ has a $2m$-factor; and that if $m$ is a nonnegative integer and $G$ is a connected graph with $|E(G)|$ even such that $d_G(v) \geq m+2$ for all vertices $v$ in $G$, then the line graph $L(G)$ has a $(2m+1)$-factor.

## 1. Introduction

In this note we deal with finite graphs without multiple edges or loops. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The line graph $L(G)$ of $G$ is the graph which has $E(G)$ as its vertex set and in which two vertices $e_i$ and $e_j$ are adjacent if and only if $e_i$ and $e_j$ have a common end vertex in $G$. The degree of a vertex $v$ of $G$ is denoted by $d_G(v)$. For a subset $A$ of $V(G)$, $G - A$ denotes the subgraph of $G$ obtained from $G$ by deleting the vertices in $A$ together with the edges incident with them. If $A$ and $B$ are disjoint subsets of $V(G)$, then we denote by $e_G(A, B)$ the number of edges of $G$ joining $A$ and $B$. A subset $A \subseteq V(G)$ is often identified with the subgraph of $G$ induced by $A$. Definitions and notations not given here may be found in [2].

For a positive integer $k$, a $k$-factor of a graph $G$ is a spanning subgraph $H$ of $G$ such that $d_H(v) = k$ for each vertex $v$ in $G$.

In [3], Chartrand, Polimeni and Stewart proved that for a connected graph $G$, a necessary and sufficient condition for $L(G)$ to have a 1-factor, is that $|E(G)|$ is even.

In this note, we consider $k$-factors with $k \geq 2$, and give a sufficient condition for the existence of such factors in the line graph of a graph $G$ in terms of the degrees of vertices of $G$.

In particular, we prove

**Theorem 1** (even factor). *Let $G$ be a graph and $m \geq 2$ be an even integer. Suppose $d_G(v) \geq m+1$ for all vertices $v$ in $G$. Then $L(G)$ has a $2m$-factor.*

As an immediate corollary to this theorem, we get

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**Corollary 2** (even factor). *Let $G$ be a graph and $m \geqslant 1$ be an odd integer, and suppose that $d_G(v) \geqslant m+2$ for all vertices $v \in V(G)$. Then $L(G)$ has a $2m$-factor.*

We also prove

**Theorem 3** (odd factor). *Let $G$ be a connected graph with $|E(G)|$ even, and $m$ be a nonnegative integer, and suppose that $d_G(v) \geqslant m+2$ for all vertices $v \in V(G)$. Then $L(G)$ has a $(2m+1)$-factor.*

Those results are the best possible in the sense that the conditions on the minimum degree can not be weakened any further. This is clear for Theorems 1 and 3. As for Corollary 2, the following example shows that the condition $d_G(v) \geqslant m+2$ can not be weakened to $d_G(v) \geqslant m+1$.

**Example.** Let $G$ be an $(m+1)$-regular graph with at least $m+2$ independent edges and $M = \{u_1v_1, \ldots, u_{m+2}v_{m+2}\}$ be a set of independent edges of $G$, where $m$ is an odd integer. Now we add two new vertices $x$ and $y$ to $G - M$, and join $x$ to $u_i$ and $y$ to $v_i$ $(1 \leqslant i \leqslant m+2)$. We denote this new graph by $G^*$. Note that in $L(G^*)$, all vertices other than the $xu_i$ and the $yv_i$ have degree exactly $2m$. So if $L(G^*)$ had a $2m$-factor $F$, then the $(m+2)$ vertices $xu_i$ would induce an $m$-regular subgraph in $F$ (and the same would be true of the $yv_i$). But this is impossible because $m$ and $m+2$ are odd. Fig. 1 shows the easiest example with $m=1$.

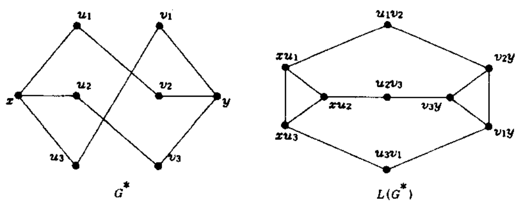


Fig. 1.

## 2. Proof of the theorems

For completeness, we first state the following well-known criterion for the existence of a $k$-factor.

**Theorem A** (Tutte [4]). *A graph $G$ has a $k$-factor if and only if*

$$\delta_G(S,T) := k\,|S| + \sum_{v \in T} d_G(v) - k\,|T| - e_G(S,T) - h(S,T) \qquad (*)$$
$$\geqslant 0$$

*for any disjoint subsets $S$, $T$ of $V(G)$, where $h(S, T)$ denotes the number of connected components $C$ of $G-(S\cup T)$ such that*

$$k\,|C| + e_G(C, T) \equiv 1 \pmod 2$$

*(such components are referred to as odd components). Furthermore, (whether $G$ has a $k$-factor or not), we have $\delta_G(S, T) \equiv k|V(G)| \pmod 2$ for any disjoint subsets $S$, $T$ of $V(G)$.*

**Proof of Theorem 1.** Let $G$ be a graph satisfying the hypothesis of Theorem 1. Let $S$ and $T$ be subsets of $V(L(G))$ such that $S\cap T=\emptyset$. For convenience, we set $U := L(G)-(S\cup T)$ and $h := h(S, T)$. We denote the sets of edges incident with $v\in V(G)$ contained in $S$, $T$ and $U$ by $S_v$, $T_v$ and $U_v$, respectively. We clearly have the following:

$$|S_v \cup T_v \cup U_v| = d_G(v) \quad \text{for each vertex } v\in V(G), \tag{1}$$

$$\sum_{v\in V(G)} |S_v| = 2|S|, \qquad \sum_{v\in V(G)} |T_v| = 2|T|, \qquad \sum_{v\in V(G)} |U_v| = 2|U|, \tag{2}$$

$$|S\cup T\cup U| = \sum_{v\in (VG)} (d_G(v))/2 = |E(G)|, \tag{3}$$

$$\sum_{x\in T} d_{L(G)}(x) = \sum_{v\in V(G)} (d_G(v)-1)|T_v|, \tag{4}$$

$$e_{L(G)}(S, T) = \sum_{v\in V(G)} |S_v|\,|T_v|. \tag{5}$$

Inserting (2), (4) and (5) into Tutte's criterion $(*)$ with $k=2m$, we obtain

$$\delta_{L(G)}(S, T) = \sum_{v\in V(G)} ((d_G(v)-m-1)|T_v| + m|S_v| - |T_v|\,|S_v|) - h.$$

Bearing this in mind, we set

$$\lambda(v) := (d_G(v)-m-1)|T_v| + m|S_v| - |T_v|\,|S_v|, \quad \text{for } v\in V(G).$$

An easy calculation shows that $\lambda(v)\geqslant 0$ for all $v\in V(G)$. For each component $X$ of $U$, let $R_X$ be the set of vertices $v\in V(G)$ such that $X\cap U_v\neq\emptyset$. Then for any two distinct components $X$, $Y$, we have $R_X\cap R_Y=\emptyset$.

So in order to prove $\delta_{L(G)}(S, T)\geqslant 0$, it suffices to show that for each *odd* component $X$ of $U$, $R_X$ contains a vertex $v$ with $\lambda(v)\geqslant 1$. Now let $X$ be an odd component of $U$. By the definition of an odd component,

$$e_{L(G)}(X, T) = \sum_{v\in R_X} |U_v|\,|T_v|$$

is odd; and we can therefore choose a vertex of $v\in R_X$ for which $|U_v|\,|T_v|$ is odd.

For this $v$, we have either $|S_v| \neq 0$ or $d_G(v) \geq m+2$; for if $|S_v| = 0$ and $d_G(v) = m+1$, then, as $m+1$ is odd, one of $|U_v|$ or $|T_v|$ is forced to be even. Since we also have $|U_v| \neq 0$ and $|T_v| \neq 0$, it is now a matter of simple arithmetic to verify that the desired inequality $\lambda(v) \geq 1$ holds. Thus $\delta_{L(G)}(S, T) \geq 0$, and the proof is complete. □

**Proof of Theorem 3.** Let $G$ be a graph satisfying the hypothesis of Theorem 3. Let $S$ and $T$ be disjoint subsets of $V(L(G))$, and define $U$, $S_v$, $T_v$, $U_v$ and $h$ as in the proof of Theorem 1. Further set

$$\lambda(v) = (d_G(v) - m - 3/2)|T_v| + (m + 1/2)|S_v| - |T_v|\,|S_v|, \quad \text{for } v \in V(G).$$

Then we have

$$\delta_{L(G)}(S, T) = \sum_{v \in V(G)} \lambda(v) - h.$$

In view of the last assertion of Theorem $A$, it suffices to show $\delta_{L(G)}(S, T) \geq -1$. Letting $\omega$ denote the number of components of $U = L(G) - (S \cup T)$, let $Y_1, \ldots, Y_\omega$ be the components of $U$ and, for each $Y_i$, let $P_i$ be the set of vertices $v \in V(G)$ such that $Y_i \cap U_v \neq \emptyset$ $(1 \leq i \leq \omega)$. Moreover we set

$$Q := V(G) - \bigcup_{1 \leq i \leq \omega} P_i = \{u_1, \ldots, u_l\} \quad \text{where } l = |Q|.$$

Thus $P_1, \ldots, P_\omega, \{u_1\}, \ldots, \{u_l\}$ are (the vertex sets of) the components of $G - (S \cup T)$. Since $G$ is connected, this means $|S \cup T| \geq \omega + l - 1$. Also again by an easy calculation, we obtain

$$\lambda(v) \geq \begin{cases} (|S_v| + |T_v|)/2 & \text{if } v \in \bigcup_{1 \leq i \leq \omega} P_i, \\ (|S_v| + |T_v|)/2 - 1 & \text{if } v \in Q. \end{cases}$$

Hence we get

$$\begin{aligned} \delta_{L(G)}(S, T) &\geq \sum_{v \in V(G)} \lambda(v) - \omega \\ &\geq \sum_{v \in V(G)} ((|S_v| + |T_v|)/2) - l - \omega \\ &= |S \cup T| - (l + \omega) \geq (l + \omega - 1) - (l + \omega) \\ &= -1. \end{aligned}$$

This completes the proof. □

## Acknowledgement

I would like to thank Professor Yoshimi Egawa for many helpful comments.

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